Homework Set 5

Due: November 18, 2008, *before class*

1. **White Dwarf Cooling**

(a) **Following the lecture notes, compute the time scale on which the *effective* surface temperature changes.**

Assume that radius of the white dwarf does not change.

$$L = 4\pi\sigma R^2 T_{\mathsf{eff}}^4$$

$\Rightarrow$

$$\frac{\mathsf{d}\ \ln L}{\mathsf{d}\ \ln T_{\mathsf{eff}}} = 4$$

Using the definitions

$$\tau_{\mathsf{cool}} = -\frac{\mathsf{d}t}{\mathsf{d}\ln L} = \frac{3}{7}\frac{\mathcal{R}}{\mu_{\mathsf{I}} {C_{\mathsf{WD}}}^{2/7}}\left(\frac{M}{L}\right)^{5/7} \approx 2.5\times10^6 \left(\frac{M}{\mathsf{M}_\odot}\right)^{5/7}\left(\frac{L}{\mathsf{L}_\odot}\right)^{-5/7}\ \mathsf{yr}$$

and analogously

$$\tau_{\mathsf{T_{eff}}} = -\frac{\mathsf{d}t}{\mathsf{d}\ln T_{\mathsf{eff}}}$$

we obtain

$$\tau_{\mathsf{T_{eff}}} = -\frac{\mathsf{d}t}{\mathsf{d}\ln T_{\mathsf{eff}}} = -\frac{\mathsf{d}\ \ln L}{\mathsf{d}\ \ln T_{\mathsf{eff}}}\frac{\mathsf{d}t}{\mathsf{d}\ln L} = 4\,\tau_{\mathsf{cool}}$$

$$\tau_{\mathsf{T_{eff}}} = \frac{12}{7}\frac{\mathcal{R}}{\mu_{\mathsf{I}} {C_{\mathsf{WD}}}^{2/7}}\left(\frac{M}{L}\right)^{5/7} \approx 10^7 \left(\frac{M}{\mathsf{M}_\odot}\right)^{5/7}\left(\frac{L}{\mathsf{L}_\odot}\right)^{-5/7}\ \mathsf{yr}$$

Score: **2**

(b) **Compute the time scale on which the core temperature changes.**

Similarly, using

$$L = M\,C_{\mathsf{WD}} T_{\mathsf{c}}^{7/2}$$

we have

$$\frac{\mathsf{d}\ \ln L}{\mathsf{d}\ \ln T_{\mathsf{c}}} = 7/2$$

and therefore

$$\tau_{\mathsf{T_c}} = -\frac{\mathsf{d}t}{\mathsf{d}\ln T_{\mathsf{c}}} = -\frac{\mathsf{d}\ \ln L}{\mathsf{d}\ \ln T_{\mathsf{c}}}\frac{\mathsf{d}t}{\mathsf{d}\ln L} = \frac{7}{2}\,\tau_{\mathsf{cool}}$$

$$\tau_{\mathsf{T_c}} = \frac{3}{2}\frac{\mathcal{R}}{\mu_{\mathsf{I}} {C_{\mathsf{WD}}}^{2/7}}\left(\frac{M}{L}\right)^{5/7} \approx 9\times10^6 \left(\frac{M}{\mathsf{M}_\odot}\right)^{5/7}\left(\frac{L}{\mathsf{L}_\odot}\right)^{-5/7}\ \mathsf{yr}$$

Score: **2**

(c) **For the case of transition to a solid, compute the cooling time scale on which the luminosity changes for the case that specific heat is increased from $\frac{3}{2}\,k_{\mathsf{B}}$/ion to $3\,k_{\mathsf{B}}$/ion. What is the signature in the WD luminosity function of such a change?**

The specific heat capacity is doubled and therefore the cooling time scale becomes twice as long. The result is a bump in the WD luminosity function. Score: **2**

(d) For the case of a cold solid, assume that the specific heat capacity per ion is $D\,T^3$ where D is a constant.

Compute the cooling time scale on which the luminosity changes for this case.

For the internal energy we then have

$$U_\mathsf{I} = \frac{D}{\mu_\mathsf{I}} M T_\mathsf{c}^3$$

Following the lecture notes step by step we get

- luminosity equals loss in internal energy

$$L = -\frac{\mathsf{d}U_\mathsf{I}}{\mathsf{d}t} = -\frac{D}{\mu_\mathsf{I}} M \frac{\mathsf{d}{T_\mathsf{c}}^3}{\mathsf{d}t} = -3\frac{D}{\mu_\mathsf{I}} M T_\mathsf{c}^2 \frac{\mathsf{d}T_\mathsf{c}}{\mathsf{d}t}$$

- using $L \propto T_\mathsf{c}^{7/2}$ (recall $L = M\,C_\mathsf{WD} T_\mathsf{c}^{7/2}$) we can write

$$L = -\frac{6}{7}\frac{D}{\mu_\mathsf{I}} M \frac{T_\mathsf{c}^3}{L}\frac{\mathsf{d}L}{\mathsf{d}t}\,, \qquad \frac{\mathsf{d}L}{\mathsf{d}t} = -\frac{7}{6}\frac{\mu_\mathsf{I}}{D}\frac{L^2}{T_\mathsf{c}^3 M}$$

- As before, define cooling time as time scale of luminosity drop by e:

$$\tau_\mathsf{cool} = -\frac{\mathsf{d}t}{\mathsf{d}\ln L} = -L\left(\frac{\mathsf{d}L}{\mathsf{d}t}\right)^{-1} = \frac{6}{7}\frac{D}{\mu_\mathsf{I}}\frac{M}{L} T_\mathsf{c}^3$$

- eliminating T_c using $L/M = C_\mathsf{WD} T_\mathsf{c}^{7/2}$ we obtain

$$\tau_\mathsf{cool} = \frac{6}{7}\frac{D}{\mu_\mathsf{I} {C_\mathsf{WD}}^{6/7}}\left(\frac{M}{L}\right)^{1/7} = \frac{6}{7}\frac{D}{\mu_\mathsf{I} C_\mathsf{WD} T_\mathsf{c}^{1/2}}$$

Without knowledge of D we cannot go any further. Note that the cooling rate now only very weakly depends on L; it is almost constant ⇒ Fast cooling! Score: **6**

2. **Neutron Stars**

Assume a non-rotating neutron star of gravitational mass $1.4\,\mathsf{M}_\odot$ and radius $10\,\mathsf{km}$.

The relativistic gravitational redshift $(1+z)$ is given by

$$1 + z = 1 \Big/ \sqrt{1 - R_\mathsf{s}/r}$$

where

$$R_\mathsf{s} = \frac{2GM}{c^2}$$

is the Schwarzschild radius. The relativistic gravitational acceleration is then given by

$$g_\mathsf{rel} = (1+z)\frac{GM}{r^2} = \frac{GM}{r^2\sqrt{1 - 2GM/(c^2\,r)}}\,.$$

(a) **Compute, in classical approximation, the gravitational potential at the surface.**

$$\phi = \frac{GM}{r} = \frac{G\,1.4\,\mathsf{M}_\odot}{10^6\,\mathsf{cm}} = \frac{6.67{\times}10^{8}\,\mathsf{cm\,s^{-2}\,g^{-1}} \times 1.4 \times 1.9891{\times}10^{33}\,\mathsf{g}}{10^6\,\mathsf{cm}}$$

$$\phi = 1.858{\times}10^{20}\,\mathsf{cm}^2\,\mathsf{s}^{-2} = 0.2067\,c^2$$

Score: **2**

(b) **Compute the relativistic gravitational potential at the surface of the neutron star.**

If something is redshifted by a factor $1+z$ when observed at infinity (large distance), this means that the original energy was higher by a factor $1+z$; whet we observes is only $1/(1+z)$ of the original energy; the remainder, $1-1/(1+z) = z/(1+z)$ has been lost. Due to energy conservation, this energy lost has to equal the potential energy, i.e., $\phi = -c^2 z/(1+z)$.

$$R_\mathsf{s} = \frac{2\,G\,M}{r^2} = \frac{2\,\times\,6.67{\times}10^{8}\,\mathsf{cm\,s^{-2}\,g^{-1}}\,\times\,1.4\,\times\,1.989{\times}10^{33}\,\mathsf{g}}{2.998{\times}10^{10}\,\mathsf{cm\,s^{-1}}} = 4.135{\times}10^{5}\,\mathsf{cm}$$

$$\frac{R_\mathsf{s}}{r} = \frac{4.135{\times}10^{5}\,\mathsf{cm}}{10^{6}\,\mathsf{cm}} = 0.4135$$

$$1+z = \frac{1}{\sqrt{1-R_\mathsf{s}/r}} = \frac{1}{\sqrt{1-0.4135}} = 1.3058\,, \quad z = 0.3058$$

$$\phi = \frac{-c^2\,z}{z+1} = -c^2\left(1-\frac{1}{z+1}\right) = -c^2\left(1-\frac{1}{1.3058}\right) = -c^2\,\times\,0.2342$$

$$\phi = -\left(2.998{\times}10^{10}\,\mathsf{cm\,s^{-1}}\right)^2\,\times\,0.2342 = -8.988{\times}10^{20}\,\mathsf{cm^2\,s^{-2}}\,\times\,0.2342$$

$$\phi = -2.105{\times}10^{20}\,\mathsf{cm^2\,s^{-2}}$$

Score: **4**

(c) **Show that for $r \gg R_\mathsf{s}$ the usual Newtonian formula for gravitational potential and acceleration is recovered.**

For $x \ll 1$ use the approximation formulae

$$\sqrt{1\pm x} \approx 1 \pm \frac{1}{2}x\,, \quad \frac{1}{x} \approx 1-x$$

We note that for $r \gg R_\mathsf{s}$ we have $R_\mathsf{s}/r \ll 1$. We then have

$$\phi = \frac{-c^2\,z}{z+1} = -c^2\left(1-\frac{1}{z+1}\right) = -c^2\left(1-\sqrt{1-\frac{R_\mathsf{s}}{r}}\right)$$

$$\phi \approx -c^2\left(1-\left(1-\frac{1}{2}\frac{R_\mathsf{s}}{r}\right)\right) = -\frac{R_\mathsf{s}c^2}{2r} = -\frac{2GMc^2}{2rc^2} = -\frac{GM}{r}$$

And for acceleration

$$1+z = \frac{1}{\sqrt{1-\frac{R_\mathsf{s}}{r}}} \approx 1 + \frac{1}{2}\frac{R_\mathsf{s}}{r} \rightarrow 1$$

$\Rightarrow$

$$g_\mathsf{rel} = (1+z)\frac{GM}{r^2} \approx \frac{GM}{r^2}$$

Score: **4**

(d) Assume the neutron star accretes material composed of solar composition. For simplicity, assume this is 75 % hydrogen and 25 % helium, by mass fraction.

Compare the specific energy that can be released by nuclear burning to $^{56}\mathsf{Fe}$ to the gravitational potential at the surface of the neutron star.

For simplicity we neglect the energy lost due to neutrinos.

mass excess of

- $^{1}\mathsf{H}$: $7.289\,\mathsf{MeV}$
- $^{4}\mathsf{He}$: $2.435\,\mathsf{MeV}$
- $^{56}\mathsf{Fe}$: $-60.601\,\mathsf{MeV}$

mass excess per nucleon:

- ^{1}H: $7.289\,\mathsf{MeV}$
- ^{4}He: $2.435\,\mathsf{MeV}/4 = 0.60875\,\mathsf{MeV}$
- ^{56}Fe: $-60.601\,\mathsf{MeV}/56 = -1.08216\,\mathsf{MeV}$

total energy release per nucleon from

- ^{1}H: $e_{^1\mathsf{H}} = 7.289\,\mathsf{MeV} - (-1.08216\,\mathsf{MeV}) = 8.3712\,\mathsf{MeV}$
- ^{4}He: $e_{^4\mathsf{He}} = 0.60875\,\mathsf{MeV} - (-1.08216\,\mathsf{MeV}) = 1.6909\,\mathsf{MeV}$

total energy per nucleon release weighed by mass fractions $X = 0.75$ and $Y = 0.25$

$$e = X \times e_{^1\mathsf{H}} + Y \times e_{^4\mathsf{He}} = 0.75 \times 8.3712\,\mathsf{MeV} + 0.25 \times 1.6909\,\mathsf{MeV} = 6.7011\,\mathsf{MeV}$$

energy release per gram is the approximately

$$e = 6.7011\,\mathsf{MeV} \times \left(\frac{1.6021773\times10^{-6}\,\mathsf{erg}}{1\,\mathsf{MeV}}\right) \times N_\mathsf{A} = 1.0736\times10^{-5}\,\mathsf{erg} \times 6.0221367\times10^{23}\,\mathsf{g}^{-1}$$

$$e = 6.4656\times10^{18}\,\mathsf{erg\,g}^{-1} = 7.1939\times10^{-3}\,c^2$$

For GR gravity:
The nuclear energy is only $e/\phi = 7.1939\times10^{-3}\,c^2/0.2342\,c^2 \approx 3.07\,\%$ of the gravitational potential energy.
For Newtonian gravity:
The nuclear energy is only $e/\phi = 7.1939\times10^{-3}\,c^2/0.2067\,c^2 \approx 3.48\,\%$ of the gravitational potential energy.

Score: **4**

(e) **If the energy released by nuclear burning was used to lift off the surface of the neutron star, how far can you lift it, or how fast can the material still move at infinity (depending on which of the energies of the previous questions is greater)?**

In principles one needs radius coordinate r as a function of potential ϕ, i.e., $r(\phi)$, and can then compute

$$\mathsf{d}r = r(\phi + e) - r(\phi)$$

. There are two possibilities:

i. Since $e \ll |\phi|$ we could use

$$\mathsf{d}r \approx -e\left(\frac{\mathsf{d}r}{\mathsf{d}\phi}\right) = -e\left(\frac{\mathsf{d}\phi}{\mathsf{d}r}\right)^{-1}$$

For Newtonian gravity:

$$\frac{\mathsf{d}\phi}{\mathsf{d}r} = \frac{\mathsf{d}}{\mathsf{d}r}\left(-\frac{GM}{r}\right) = \frac{GM}{r^2}$$

and we obtain

$$\mathsf{d}r \approx \frac{er^2}{GM} = \frac{2er^2}{c^2\,R_\mathsf{s}} = \frac{2 \times 7.1939\times10^{-3} \times \left(10^6\,\mathsf{cm}\right)^2}{4.135\times10^5\,\mathsf{cm}} = 34796\,\mathsf{cm}$$

$$\mathsf{d}r \approx 350\,\mathsf{m}$$

For GR gravity:

$$\frac{\mathsf{d}\phi}{\mathsf{d}r} = -c^2\frac{\mathsf{d}}{\mathsf{d}r}\left(1 - \sqrt{1 - R_\mathsf{s}/r}\right) = \frac{c^2\,R_\mathsf{s}}{2\,r^2}\left(1 - \frac{Rs}{r}\right)^{-1/2}$$

and we obtain

$$\mathsf{d}r \approx \frac{2\,e\,r^2}{c^2\,R_\mathsf{s}}\left(1 - \frac{R_\mathsf{s}}{r}\right)^{1/2} = \frac{2 \times 7.1939\times10^{-3} \times \left(10^6\,\mathsf{cm}\right)^2}{4.135\times10^5\,\mathsf{cm}}(1 - 0.4135)^{1/2} = 26647\,\mathsf{cm}$$

$$\mathsf{d}r \approx 270\,\mathsf{m}$$

ii. Exact solution.

For Newtonian gravity:

$$e = GM\left(\frac{1}{r} - \frac{1}{r + \mathsf{d}r}\right)$$

$$\mathsf{d}r = r\left(\frac{1}{1 - er/GM} - 1\right) = r\left(\frac{1}{1 - 2er/R_{\mathsf{s}}c^2} - 1\right)$$

$$\frac{2er}{R_{\mathsf{s}}c^2} = \frac{2 \times 7.1939{\times}10^{-3} \times 10^6\,\mathsf{cm}}{4.135{\times}10^5\,\mathsf{cm}} = 3.4796{\times}10^{-2}$$

$$\mathsf{d}r = 10^6\,\mathsf{cm} \times \left(\frac{1}{1 - 3.4796{\times}10^{-2}} - 1\right) = 36050\,\mathsf{cm} \approx 360\,\mathsf{m}$$

For GR gravity:

$$e = -c^2\left(1 - \sqrt{1 - R_{\mathsf{s}}/r} - \left(1 - \sqrt{1 - R_{\mathsf{s}}/(r + \mathsf{d}r)}\right)\right)$$

$$e = c^2\left(\sqrt{1 - R_{\mathsf{s}}/(r + \mathsf{d}r)} - \sqrt{1 - R_{\mathsf{s}}/r}\right)$$

$$\mathsf{d}r = R_{\mathsf{s}}\left[1 - \left(\sqrt{1 - R_{\mathsf{s}}/r} + \frac{e}{c^2}\right)^2\right]^{-1} - r = R_{\mathsf{s}}\left[1 - \left(\frac{1}{1+z} + \frac{e}{c^2}\right)^2\right]^{-1} - r$$

$$\mathsf{d}r = 413492\,\mathsf{cm}\left[1 - \left(\frac{1}{1.3057} + 7.1939{\times}10^{-3}\right)^2\right]^{-1} - 10^6\,\mathsf{cm} = 27647\,\mathsf{cm} \approx 280\,\mathsf{m}$$

Score: **4**

3. **Black Holes**

(a) **Compute the Schwarzschild Radius of the sun and of the earth.**

$$R_{\mathsf{s}} = \frac{2GM}{c^2} = M \times \frac{2G}{c^2} = M \times \frac{2 \times 6.67{\times}10^{-8}\,\mathsf{cm\,s^2\,g^{-1}}}{(2.9979{\times}10^{10}\mathsf{cm\,s^{-1}})} = M \times 1.4842{\times}10^{-28}\mathsf{cm\,g^{-1}}$$

$$R_{\mathsf{s},\odot} = \mathsf{M}_\odot \times \frac{2G}{c^2} = 1.9891{\times}10^{33}\,\mathsf{g} \times 1.4842{\times}10^{-28}\mathsf{cm\,g^{-1}} = 2.95{\times}10^5\,\mathsf{cm} \approx 3\,\mathsf{km}$$

$$R_{\mathsf{s}} = 2.954{\times}10^5\,\mathsf{cm}\left(\frac{M}{\mathsf{M}_\odot}\right)$$

$$R_{\mathsf{s},\oplus} = M_\oplus \times \frac{2G}{c^2} = 5.9742{\times}10^{27}\,\mathsf{g} \times 1.4842{\times}10^{-28}\mathsf{cm\,g^{-1}} = 0.8867\,\mathsf{cm}$$

Score: **4**

(b) **In classical approximation using Keplerian mechanics, at what orbital distance would an object be from the center of the Sun assumed as a point mass that has an orbital velocity of the speed of light?**

In Keplerian orbit centrifugal force equals gravity:

$$\frac{G\mathsf{M}_\odot}{r^2} = \frac{v^2}{r}$$

for $v = c$ one obtains for r:

$$r = \frac{G\mathsf{M}_\odot}{c^2} = \frac{1}{2}R_{\mathsf{s},\odot} = 1.5{\times}10^5\,\mathsf{cm} = 1.5\,\mathsf{km}$$

Score: **4**

Please use cgs units for calculations and numerical values.