Astrophysics I: Stars and Stellar Evolution

AST 4001

Alexander Heger[1,2,3]

[1]School of Physics and Astronomy
University of Minnesota

[2]Nuclear & Particle Physics, Astrophysics & Cosmology Group, T-2
Los Alamos National Laboratory

[3]Department of Astronomy and Astrophysics
University of California at Santa Cruz

Stars and Stellar Evolution, Fall 2008

Overview

1. Recap
   - Atmospheric Temperature stratification
   - Gray Atmosphere
2. Stellar Atmospheres
   - Eddington-Barbier Relation
   - Limb Darkening
3. Excited Atoms and Ionization
   - Line Formation

Summary of Temperature in Atmosphere

- atmosphere in local thermodynamic equilibrium
  Kirchhoff's law applies: $j_\nu = \kappa_\nu B_\nu(T)$
- plane parallel atmosphere with a thickness much smaller than radius of star
  curvature can be neglected
- assume gray atmosphere, i.e., a suitable average absorption coefficient $\bar{\kappa}$ can be found so that all quantities can be integrated
- the atmosphere is in radiative equilibrium
  no net energy is generated or absorbed (consumed).

Radiative Transfer in Atmosphere

- compute the first moment of the transfer equation

$$\cos\Theta\,\frac{\mathrm{d}I_\nu(z,\Theta)}{\mathrm{d}\tau_\nu}=I_\nu(z,\Theta)-B_\nu(T)$$

  by multiplication with $\cos\Theta$ and integration over all solid angles

$$\oint_{4\pi}\cos^2\Theta\,\frac{\mathrm{d}I_\nu(z,\Theta)}{\mathrm{d}\tau_\nu}\,\mathrm{d}\omega=\frac{\mathrm{d}}{\mathrm{d}\tau_\nu}\oint_{4\pi}\cos^2\Theta\,I_\nu(z,\Theta)\,\mathrm{d}\omega=\ldots$$

$$\ldots=\oint_{4\pi}\cos\Theta\,I_\nu(z,\Theta)\,\mathrm{d}\omega-\oint_{4\pi}\cos\Theta\,B_\nu(T)\,\mathrm{d}\omega$$

- Note that because $B_\nu(T)$ is isotropic, the last term vanishes and we obtain

$$\frac{\mathrm{d}K_\nu(z)}{\mathrm{d}\tau_\nu}=\frac{F_\nu(z)}{4\pi}$$

Radiative Transfer in Gray Atmosphere

- define mean opacity $\bar{\kappa}$ such that we obtain a mean optical depth τ by

$$\mathrm{d}\tau = \bar{\kappa}\,\mathrm{d}z$$

- The frequency integral of the first moment of the transfer equation,

$$\int_0^\infty \frac{\mathrm{d}K_\nu(z)}{\mathrm{d}\tau_\nu}\,\mathrm{d}\nu = \int_0^\infty \frac{F_\nu(z)}{4\pi}\,\mathrm{d}\nu$$

  then becomes

$$\frac{\mathrm{d}K(z)}{\mathrm{d}\tau} = \frac{F(z)}{4\pi} = \frac{F}{4\pi}$$

Radiative Transfer in Gray Atmosphere

- differentiation with regard to τ yields in radiative equilibrium

$$\frac{\mathrm{d}^2 K(z)}{\mathrm{d}\tau^2} = \frac{1}{4\pi}\frac{\mathrm{d}F}{\mathrm{d}\tau} = J - B = 0$$

- where J and B are now frequency-integrated quantities.
- to evaluate K we will assume that I is isotropic; since we only multiply it with a positive quantity, $\cos^2\Theta$, there will be no effect from almost, but not quite, cancellation of two large quantities (at top and bottom) as it is in the case of the flux
- we hence can approximate from the definition of K

$$K = \frac{1}{4\pi}\oint_{4\pi} I\,\cos^2\Theta\,\mathrm{d}\omega = \frac{1}{2}J\int_0^{\pi}\cos^2\Theta\,\sin\Theta\,\mathrm{d}\Theta = \frac{1}{3}J$$

(Eddington approximation) [yet another]

Radiative Transfer in Gray Atmosphere

- we can now use $K = \frac{1}{3}J$ in the first moment of the transfer equation,

$$\frac{\mathrm{d}K(z)}{\mathrm{d}\tau} = \frac{F}{4\pi}$$

and obtain

$$\frac{\mathrm{d}J(z)}{\mathrm{d}\tau} = \frac{3}{4\pi}F$$

- integration with regards to τ then gives

$$J = \frac{3}{4\pi}F\,\tau + \mathrm{const.}$$

- But we also have from the definition of T_{eff}, which is considered to be a constant:

$$J = B = \frac{\sigma}{\pi}T^4 = \frac{3}{4}\frac{\sigma}{\pi}T_{\mathrm{eff}}^4(\tau + c_3)$$

Radiative Transfer in Gray Atmosphere

- to derive the constant c_3 we consider that at the surface, $\tau = 0$, there is no inward flux, but we assume that in our approximation the intensity at the surface is independent of direction. Let us call the intensity at the surface I_0^+.
- at the surface we then have

$$J(0) = \frac{2\pi}{4\pi} \int_0^{\pi/2} I(0) \sin\Theta \, \mathrm{d}\Theta = \frac{I_0^+}{2}$$

$$F(0) = 2\pi \int_0^{\pi/2} I(0) \cos\Theta \sin\Theta \, \mathrm{d}\Theta = \pi I_0^+$$

and hence

$$J(0) = \frac{F(0)}{2\pi} = B(0) = \frac{\sigma T^4(0)}{\pi} = \frac{1}{2\pi} \sigma T_{\mathrm{eff}}^4$$

Radiative Transfer in Gray Atmosphere

- therefore, at the surface we now have

$$T^4(0) = \frac{1}{2} T_{\text{eff}}^4$$

- from

$$\frac{\sigma}{\pi} T^4 = \frac{3}{4} \frac{\sigma}{\pi} T_{\text{eff}}^4 (\tau + c_3)$$

we obtain

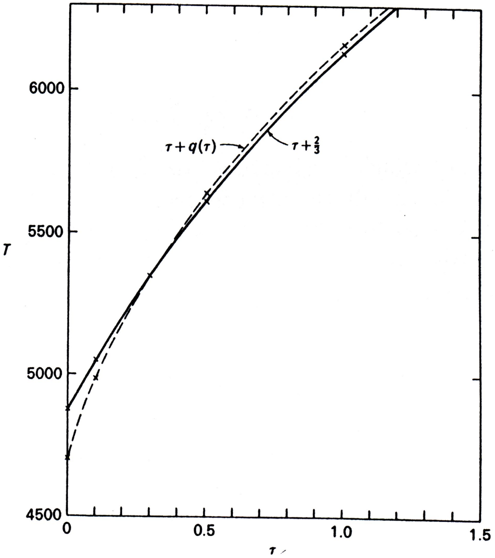
$$\frac{1}{2} T_{\text{eff}}^4 = \frac{3}{4} T_{\text{eff}}^4 \, c_3 \quad \Rightarrow \quad c_3 = \frac{2}{3}$$

- and the final distribution of temperature in a gray atmosphere is

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3} \right)$$

- Note that $T = T_{\text{eff}}$ at $\tau = 2/3$

Approximation Compared with True Stratification



more generally we can write

$$T^4 = \frac{3}{4} T_{\rm eff}^4 \left(\tau + q(\tau)\right)$$

Quiz

Compute the moments of I: J, F, and K

- in the center of the star
- in thermodynamic equilibrium (TE)

Instructions:

- Work on this yourself and write down your solution (2 min)
- Discuss in groups of 2-3 (2 min)

Overview

1. Recap
   - Atmospheric Temperature stratification
   - Gray Atmosphere
2. Stellar Atmospheres
   - Eddington-Barbier Relation
   - Limb Darkening
3. Excited Atoms and Ionization
   - Line Formation

Eddington-Barbier Relation

- at surface of star ($\tau = 0$) we divide flux in *inward* ($\Theta > \pi/2$) and *outward* ($\Theta < \pi/2$) parts
- the inward part is zero and we have

$$F_\nu(0) = 2\pi \int_0^1 I_\nu(0, \Theta)\, \mu \, \mathrm{d}\mu$$

- recall: in the "gray" case we had linear relation between T^4 and τ. Now we try an analogous approximation for frequency-dependent transport:

$$B_\nu = a_\nu + b_\nu \, \tau$$

with constants a_ν and b_ν.

Eddington-Barbier Relation

- multiplying the transfer equation,

$$\cos\Theta \frac{\mathrm{d}I_\nu(z,\Theta)}{\mathrm{d}\tau_\nu} = I_\nu(z,\Theta) - B_\nu(T)$$

by $e^{-\tau_\nu \sec\Theta}$ we have

$$\frac{\mathrm{d}I_\nu(z,\Theta)}{\sec\Theta\,\mathrm{d}\tau_\nu}\, e^{-\tau_\nu \sec\Theta} = \left[I_\nu(z,\Theta) - B_\nu(T)\right] e^{-\tau_\nu \sec\Theta}$$

- and can re-write this in the form

$$\frac{\mathrm{d}\left(I_\nu(z,\Theta)\, e^{-\tau_\nu \sec\Theta}\right)}{\mathrm{d}(\sec\Theta\,\tau_\nu)} = -B_\nu(T)\, e^{-\tau_\nu \sec\Theta}$$

- and integrated to obtain the emergent intensity at the top

$$\left[I_\nu(z,\Theta)\, e^{-\tau_\nu \sec\Theta}\right]_0^\infty = -I_\nu(0,\Theta) = -\int_0^\infty B_\nu(T)\, e^{-\tau_\nu \sec\Theta} \mathrm{d}(\sec\Theta\,\tau_\nu)$$

Eddington-Barbier Relation

- now using $B_\nu = a_\nu + b_\nu\,\tau$ we can write

$$I_\nu(0,\Theta) = \int_0^\infty a_\nu\, e^{-\tau_\nu \sec\Theta}\, \mathrm{d}(\sec\Theta\,\tau_\nu) + b_\nu \int_0^\infty \tau_\nu\, e^{-\tau_\nu \sec\Theta}\, \mathrm{d}(\sec\Theta\,\tau_\nu)$$

- using

$$\int_0^\infty e^{-ax}\,\mathrm{d}x = \frac{1}{a}\,, \quad \int_0^\infty x\, e^{-ax}\,\mathrm{d}x = \frac{1}{a^2}$$

we obtain

$$I_\nu(0,\Theta) = a_\nu + b_\nu\, \cos\Theta = B_\nu(\tau_\nu = \cos\Theta)$$

(First Eddington-Barbier Relation)

Eddington-Barbier Relation

- if we use this result in

$$F_\nu(0) = 2\pi \int_0^1 I_\nu(0,\Theta)\,\mu\,\mathrm{d}\mu$$

we obtain

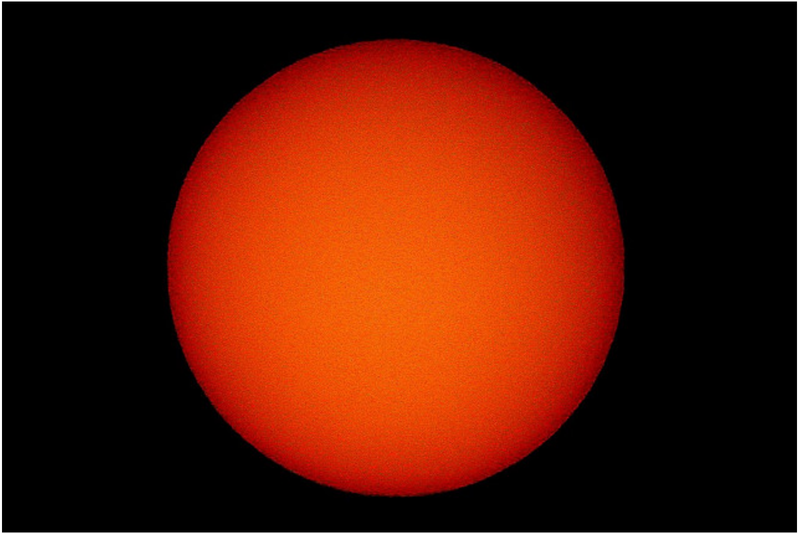
$$F_\nu(0) = 2\pi \int_0^1 (a_\nu + b_\nu \cos\Theta)\,\cos\Theta\,\mathrm{d}(\cos\Theta) = \left(a_\nu + \frac{2}{3} b_\nu\right)\pi$$

$$F_\nu(0) = \pi\, B_\nu\left(\tau = \frac{2}{3}\right)$$

(Second Eddington-Barbier Relation)

- $\Rightarrow$ flux from stellar surface at a particular frequency is determined by Planck function at $T(\tau_\nu = 2/3)$

The Sun



Limb Darkening

- using the same analysis for the "gray" case, we would use the approximation

$$B(\tau) = a + b\,\tau.$$

- we know that

$$B = \frac{\sigma}{\pi}\,T^4 = \frac{3}{4\pi}\,F\left(\tau + \frac{2}{3}\right)$$

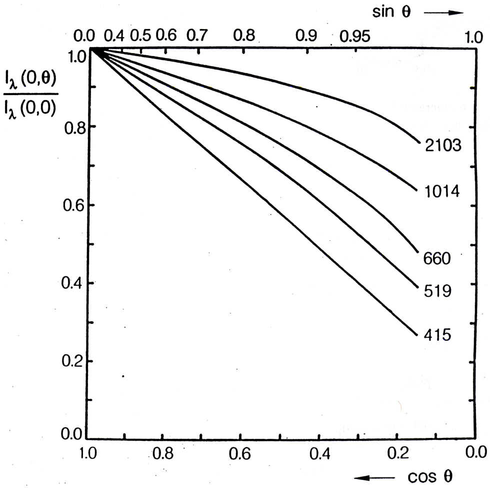
which gives $a = F/(2\pi)$, $b = 3F/(4\pi)$

- using the expression $I(0,\Theta) = a + b\cos\Theta$ we obtain intensity as a function of angle (Limb Darkening)

$$I(0,\Theta) = \frac{F}{4\pi}(2 + 3\cos\Theta)$$

- at the limb of the sun ($\Theta = \pi/2$) one should see only 40 % of the intensity at the center of the solar disk ($\Theta = 0$)

Limb Darkening



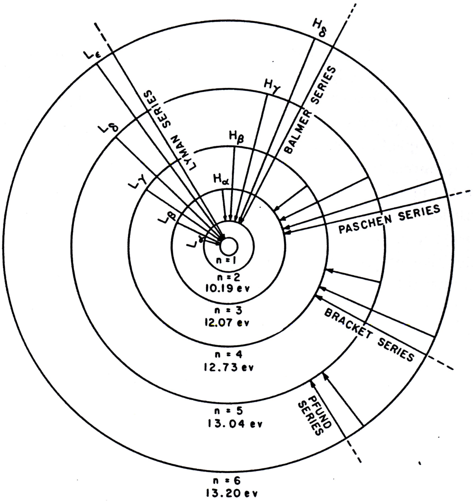
limb darkening for different wave lengths (in nm)

Note that the intensity integrate over all wave lengths fulfills formula to within a few percent in range $\cos\Theta = 1 \ldots 0.1$.

Overview

1. Recap
   - Atmospheric Temperature stratification
   - Gray Atmosphere
2. Stellar Atmospheres
   - Eddington-Barbier Relation
   - Limb Darkening
3. Excited Atoms and Ionization
   - Line Formation

Hydrogen level scheme



Saha Function - Levels

ratio of occupation N_i of levels $i = n$ and $i = n'$:

$$\frac{N_n}{N_{n'}} = \frac{g_n}{g_{n'}} \exp\left[-(\chi_n - \chi_{n'})/kT\right] , \quad g_n = 2J + 1 , \quad J = L + S$$

g_n is called the *statistical weight*,
J, L, and S are total and orbital angular momentum and spin of the electron

partition function and total number of atoms:

$$u(T) = \sum g_n \exp\left(-\chi_n/kT\right) , \quad N = \sum_{n=1,\ldots} N_n$$

Using $\Theta = 5060/T$ in eV we can write:

$$\frac{N_n}{N} = \frac{g_n}{u(T)} 10^{-\Theta\chi_n} ,$$

Saha Function - Ionization

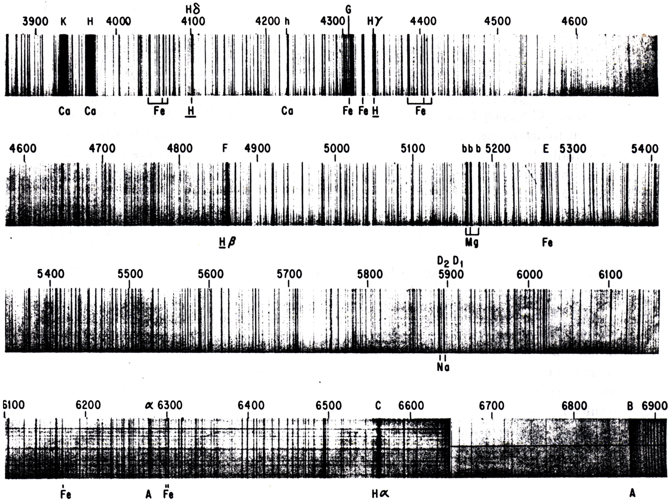
Similarly, using

$$P_\mathrm{e} = n_\mathrm{e} k_\mathrm{B} T$$

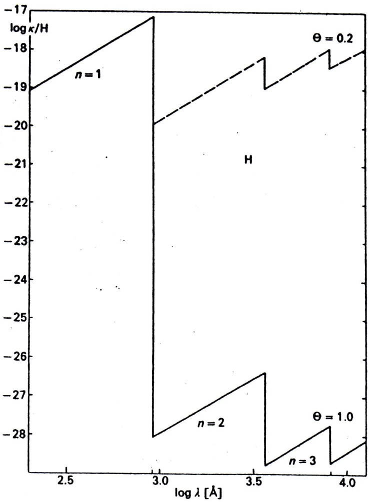
we can also write the ratio of ionization levels r as

$$\log\left(\frac{N_{r+1}}{N_r} P_\mathrm{e}\right) = \Theta\chi_r + 2.5 \log T - \log \frac{2u_{r+1}}{u_r} - 1.48$$

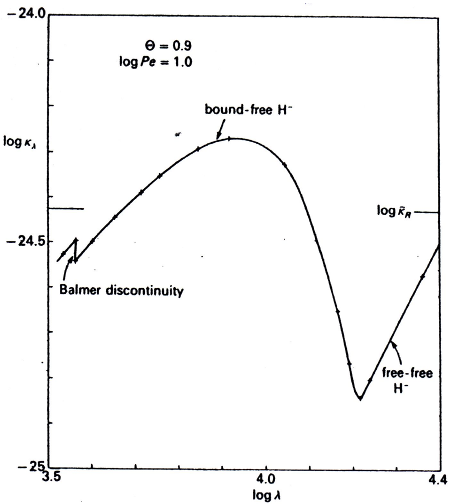
Solar Spectrum



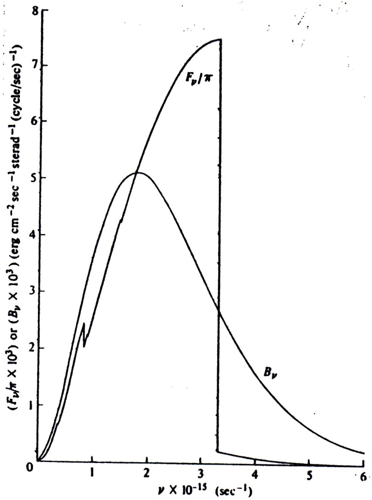
Absorption coefficient per hydrogen atom



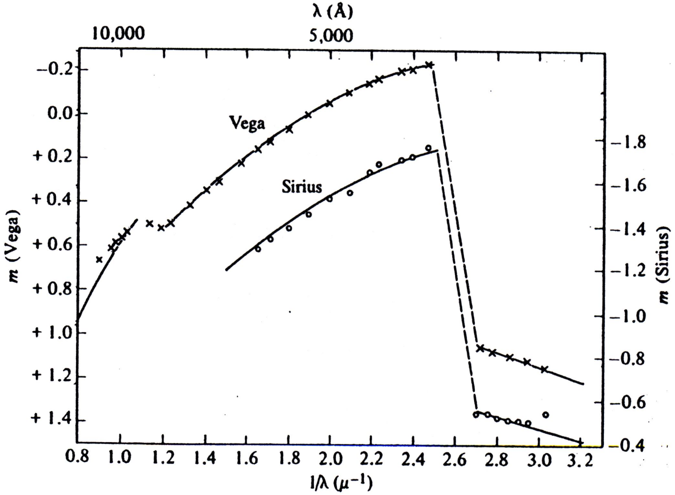
Continuous Absorption coefficient per hydrogen atom



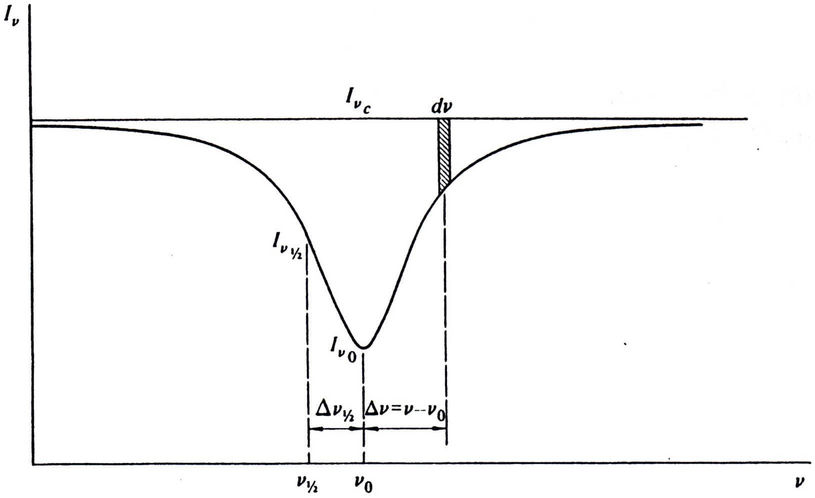
Emergent Flux, Comparison Model Atmosphere - Black Body



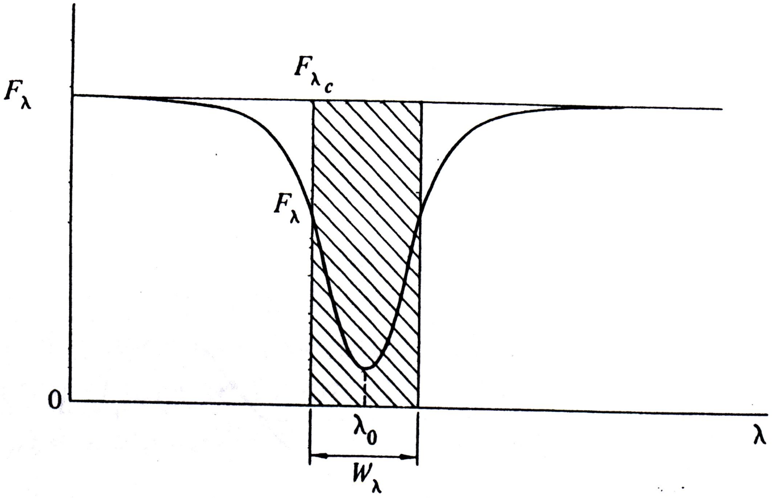
Sirius and Vega observed and model



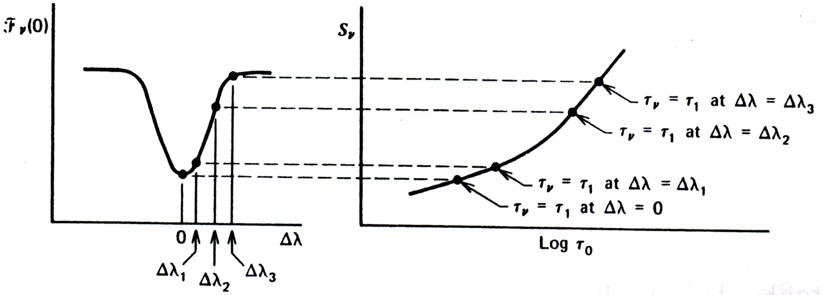
Line Profile



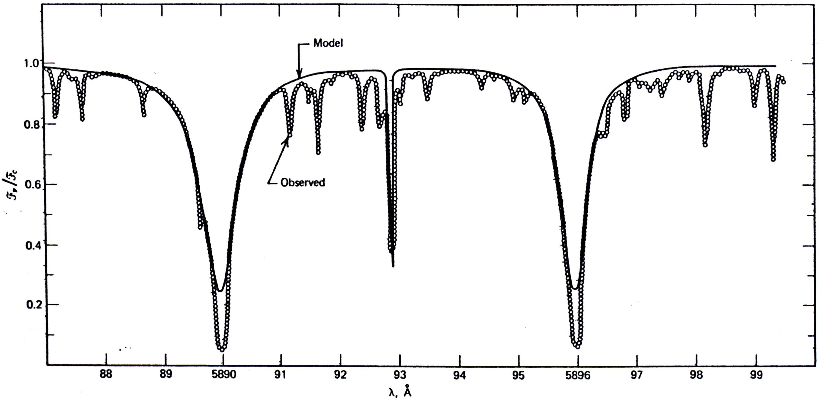
Equivalent Width



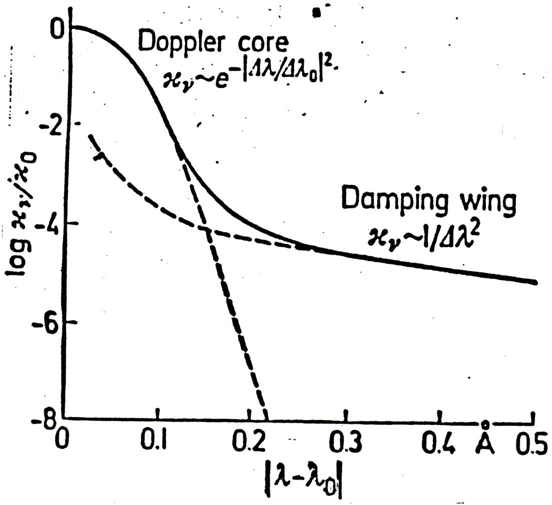
Line Formation



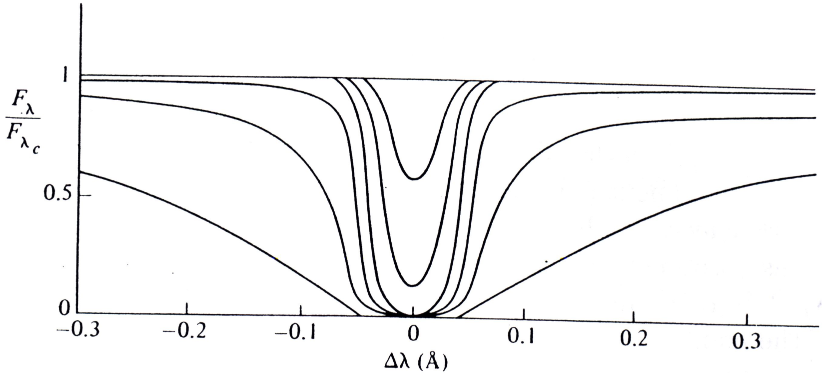
Sodium D Line



Line Broadening



CaII K line, Theoretical Models



Schematic curve of growth