Mid-Term I

Time: 45 min

Physical and Astronomical Constants

CODATA Internationally recommended values of the Fundamental Physical Constants.
The most recent values can be found at `http://physics.nist.gov/cuu/Constants`.

Solar mass	$M_\odot$	=	$1.989 \cdot 10^{33}$ gm
Solar radius	$R_\odot$	=	$6.955 \cdot 10^{10}$ cm
Solar effective temperature	$T_{\mathrm{eff},\odot}$	=	5780 K
Solar surface gravity	$g_{s,\odot}$	=	$2.744 \cdot 10^{4}$ cm/sec^2
Solar luminosity	$L_\odot$	=	$3.846 \cdot 10^{33}$ erg/sec
Solar absolute bol. mag.	$M_{b,\odot}$	=	$+4.77$
Velocity of light in vacuo	c	=	$2.99792458 \cdot 10^{10}$ cm/sec
Constant of gravitation	G	=	$6.6742 \cdot 10^{-8}$ cm^3/(gm s^2)
Boltzmann constant	k	=	$1.3807 \cdot 10^{-16}$ erg/K
Avogadro's number	N_0	=	$6.022 \cdot 10^{23}$ mole^{-1}
Atomic mass unit	1 AMU	=	$1/N_0 = H$
		=	$1.66054 \cdot 10^{-24}$ gm $= 931.5$ MeV
Gas constant	$\mathcal{R}$	=	$8.314 \cdot 10^{7}$ erg/K/mole
Planck's constant	h	=	$6.626 \cdot 10^{-27}$ erg sec
	$\hbar = h/2\pi$	=	$1.0546 \cdot 10^{-27}$ erg sec
Electronic charge	e	=	$4.803 \cdot 10^{-10}$ e.s.u.
		=	$1.602 \cdot 10^{-19}$ C
Fine structure constant	$e^2/\hbar c$	=	$1/137.036$
Stefan-Boltzmann constant	σ	=	$5.670 \cdot 10^{-5}$ erg/(cm^2 K^4 sec)
Radiation pressure constant	$a = 4\sigma/c$	=	$7.566 \cdot 10^{-15}$ erg/(cm^3 K^4)
Electron rest mass	m_e	=	$9.109 \cdot 10^{-28}$ gm $= 0.5110$ MeV
Mass ratio proton/electron	m_p/m_e	=	1836.2
Mass of hydrogen atom	H^1	=	$1.6734 \cdot 10^{-24}$ gm
		=	1.0081 AMU
Classical electron radius	$e^2/m_e c^2$	=	$2.818 \cdot 10^{-13}$ cm
Compton wavelength of electron	$\lambda_C = \hbar/m_e c$	=	$3.8616 \cdot 10^{-11}$ cm
Thomson scattering cross section	σ_0	=	$(8\pi/3)\left(e^2/m_e c^2\right)^2$
		=	$0.6652 \cdot 10^{-24}$ cm^2
Electron volt	1 eV	=	$1.602 \cdot 10^{-12}$ erg $= 11604$ K

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1. **Stability of Stars**

   (a) For a star in hydrostatic equilibrium, two of the stellar structure equations are

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}$$

   Use these two equations to express local the pressure scale height H_P,

$$H_p = -\frac{\mathsf{d}r}{\mathsf{d}\ln P}$$

   in terms of P, ρ and $g = Gm/r^2$ in a form that does not contain derivatives.

$$H_p = -\frac{\mathsf{d}r}{\mathsf{d}\ln P} = -P\,\frac{\partial r}{\partial m}\bigg/\frac{\partial P}{\partial m} = -P\frac{1}{4\pi r^2\rho}\left(-\frac{4\pi r^4}{Gm}\right) = \frac{P\,r^2}{\rho Gm} = \frac{P}{g\rho}$$

   Score: **2**

   (b) Consider the definition of ∇_μ,

$$\nabla_\mu = \frac{\mathsf{d}\ \ln\mu}{\mathsf{d}\ \ln P}\,.$$

   What is the quantity "μ"?

   μ is the mean molecular weight.

   Score: **1**

   (c) **In a star in hydrostatic equilibrium, if $\nabla_\mu > 0$, does μ increase or decrease with increasing radius (or mass) coordinate?**

   Hint: Consider how pressure changes with radius.

   For this condition, μ does decrease with increasing radius coordinate.

   Score: **2**

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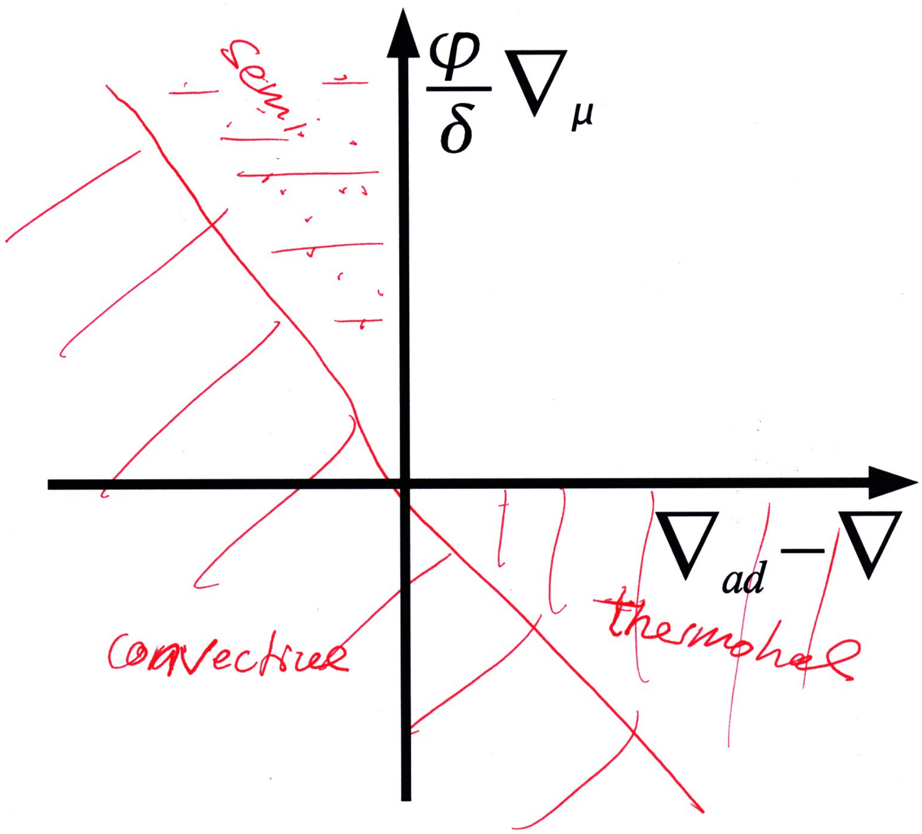
(d) The condition for *stability* against convection is given by the Ledoux criterion:

$$\nabla_{\mathsf{rad}} < \nabla_{\mathsf{ad}} + \frac{\varphi}{\delta}\nabla_\mu$$

In the figure below indicate regions of convection.



Recall the definitions from the equation of state:

$$\nabla = \left(\frac{\mathsf{d}\ \ln T}{\mathsf{d}\ \ln P}\right)_{\mathsf{star}}, \quad \nabla_{\mathsf{ad}} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_{\mathsf{ad}}, \quad \delta = -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_{P,\mu}, \quad \phi = \left(\frac{\partial \ln \rho}{\partial \ln \mu}\right)_{P,T}$$

and that "ad" means "at constant entropy". ∇ is the actual temperature gradient in the star with respect to pressure.

Score: **3**

(e) **Bonus: Indicate semiconvective and thermohaline regions as well.**

Score: **extra points: 2**

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2. **Burning in Stars**

(a) **List the six major nuclear burning stages in massive stars.**

i. hydrogen burning
ii. helium burning
iii. carbon burning
iv. neon burning
v. oxygen burning
vi. silicon (sulfur) burning

Score: **6**

(b) The first of these two major burning stages of massive stars last much longer than the last four.
What is the main reason for that?
In the last four stages neutrinos carry away most of the energy directly from the interior of the star.

Score: **2**

(c) Two major different modes (cycles, chains) of hydrogen burning exist in massive stars.
Which are they?

i. CNO cycle
ii. pp chains

Score: **2**

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(d) The Eddington luminosity of a star is given by

$$L_{\mathrm{edd}} = 3.3 \times 10^4 \left(\frac{M}{\mathrm{M}_\odot}\right) \mathrm{L}_\odot$$

How long can a $10\,\mathrm{M}_\odot$ star entirely composed of hydrogen shine at Eddington luminosity if it burns its entire fuel (hydrogen) to helium?

Assume burning releases $27\,\mathrm{MeV}$ per helium nucleus formed,
$\mathrm{L}_\odot = 3.83\times10^{33}\,\mathrm{erg\,s^{-1}}$
$\mathrm{M}_\odot = 1.9891\times10^{33}\,\mathrm{g}$
$N_\mathrm{A} = 6.022\times10^{23}\,\mathrm{mol^{-1}}$
$1\,\mathrm{eV} = 1.6\times10^{-12}\,\mathrm{erg}$

The time scale can be computed by dividing the total energy supply by the luminosity:

$$\tau = \frac{E}{L} = \frac{M \times 27/4\,\mathrm{MeV} \times N_\mathrm{A} \times 1.6\times10^{-12}\mathrm{eV/erg} \times 10^6\,\mathrm{MeV/eV}}{M \times 3.3\times10^4\,\mathrm{L}_\odot/\mathrm{M}_\odot}$$

$$\tau = 1.0\times10^{14}\,\mathrm{s} = 3.25\times10^6\,\mathrm{yr}$$

A star made of pure hydrogen can shine for about $3\,\mathrm{Myr}$ at its Eddington luminosity.

Score: **4**

(e) **Under the same assumptions, how long can a $100\,\mathrm{M}_\odot$ star entirely composed of hydrogen shine at Eddington luminosity if it burns its entire fuel (hydrogen) to helium?**

The same time.
Explanation:
Both, the fuel supply and the Eddington luminosity scale as mass, so their ratio remains constant.

Score: **2**

(f) **Still under the same assumptions, what is the exponent "ν" in the mass-luminosity relation**

$$L \propto M^\nu$$

for such stars?

Luminosity is proportional to mass, i.e., $\nu = 1$

Score: **2**

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3. **Equation of state.**

   (a) **List (name) the four major regimes of the equation of state in stars.**

   i. radiation dominated ($P = \frac{a}{3}T^4$), $\gamma_{\mathsf{ad}} = 4/3$
   ii. ideal gas ($P = \frac{\mathcal{R}\mathcal{T}\rho}{\mu}$) $\gamma_{\mathsf{ad}} = 5/3$
   iii. non-relativistic degenerate gas, $\gamma_{\mathsf{ad}} = 5/3$
   iv. relativistic degenerate gas, $\gamma_{\mathsf{ad}} = 4/3$

   Score: **4**

   (b) **BONUS: Can you assign an adiabatic index γ_{ad} to each of these equations of state?**

   Hint: Recall the definition of γ_{ad}:

$$\gamma_{\mathsf{ad}} = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{\mathsf{ad}}$$

   Score: **extra points: 2**

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4. **Virial Theorem and Equation of State**

   (a) **For a star in hydrostatic equilibrium composed of ideal gas, is its total energy (internal energy of gas + gravitational energy + kinetic energy) positive or negative?**

   It is negative.

   Score: **1**

   (b) Assume a star of luminosity $L > 0$ in hydrostatic and thermal equilibrium that has used up all its internal nuclear energy, but continues to shine.

   Will its total energy increase or decrease? Will the internal energy of the gas increase or decrease? How will its average temperature change?

   The total energy will decrease, the star will contract, its total internal energy will increase, and the average temperature will increase.
   This is a result of the virial theorem.

   Score: **4**

   (c) Now consider the same situation as above, but for a star of completely non-relativistic degenerate gas.

   How does pressure depend on temperature for non-relativistic degenerate gas? How will the average temperature of this star change?

   For non-relativistic degenerate gas pressure is independent of temperature. Therefore, when the star looses energy – cools – it will not contract, it will just cool down. The average temperature of the star will decrease.

   Score: **2**